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Bound states in the three-dimensional ϕ^4 model.

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We discuss the spectrum of the three dimensional ϕ^4 theory in the broken symmetry phase. In this phase the effective potential between the elementary quanta of the model is attractive and bound states of two or more of them may exist. We give theoretical and numerical evidence for the existence of these bound states. Looking in particular at the Ising model realization of the ϕ^4 theory we show, by using duality, that these bound states are in one-to-one correspondence with the glueball states of the gauge Ising model. We discuss some interesting consequences of this identification.

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Three dimensional statistical systems with global Z_2 symmetry, the Ising model being the classic example, lie in the universality class of the ϕ^4 field theory. Critical phenomena in such systems are known to be accurately described by simple perturbative methods [1]. Given the success of perturbative methods, the appearance of excited states in the broken symmetry phase of the critical Ising model and in the 3D ϕ^4 theory, which were found in [2,3], comes as a surprise, since scalar field theory apparently describes only one particle as long as interactions can be treated perturbatively. We shall argue that this is not the case and there is room for a rich spectrum of excitations in the broken symmetry phase of the ϕ^4 theory even if the interaction is weak.

The excited states show up as poles of the correlation functions in the complex momentum plane and give visible contribution to certain universal quantities. The first excited state lies just below the two-particle threshold: its mass is $M = 1.83(3)m$ [2], where m is the mass gap. The closeness of M to the threshold suggests the interpretation of this excitation as a weakly coupled bound state of two elementary excitations.

Indeed, the two-particle forces are attractive in the broken-symmetry phase of the ϕ^4 theory, and bound states of two or more elementary quanta may in principle be formed. In four dimensions, these states indeed exist in the low-temperature regime, but disappear as the continuum limit is approached [4,5], in agreement with triviality.

In this letter we address the three-dimensional case. Numerical simulations show that non-perturbative states survive the continuum limit in 3d [2,3]. We shall argue that these states can be identified with the multiparticle bound states. By considering the Ising realization of the ϕ^4 model and using duality we shall also show that there is an exact one-to-one mapping between the bound states of the Ising model (and hence, thanks to universality, also of the ϕ^4 theory) and the glueball states of the gauge Ising model.

Bound states in the ϕ^4 theory. We consider the ϕ^4 theory:

$$S = \int d^3x \left[\frac{1}{2} (\partial\phi)^2 + \lambda(\phi^2 - v^2)^2 \right].$$

The field $\sigma = \phi - v$ acquires the mass $m^2 = 8\lambda v^2$ at the tree level and is reasonably weakly coupled in the critical regime [1], since the critical value of the dimensionless interaction constant, λ/m, is not too big. The forces between elementary quanta of the field σ are attractive: This can be shown by inspecting the scattering of two non-relativistic particles. There are three leading-order diagrams contributing to this process, shown in Fig. 1. The contact interaction, diagram (a), contributes -12λ

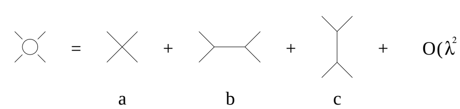

FIG. 1. Feynman diagrams contributing to the $2 \to 2$ amplitude at the leading order in λ.

to the scattering amplitude. As a first approximation, we can neglect altogether the momentum flow in diagrams (b) and (c). In this way, diagrams (b) and (c) contribute -12λ and 72λ, respectively. Collecting the three terms together, we get for the amplitude $\mathcal{A} = 48\lambda$. The positive sign of the amplitude means that the particles attract each other.

In this limit, the non-relativistic Hamiltonian describing the interaction of two particles is

$$H = \frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} - \frac{12\lambda}{m^2}\,\delta(\mathbf{x}_1 - \mathbf{x}_2), \qquad (1)$$

which reproduces the field-theory scattering amplitude in the Born approximation [6]. Note the factor of $1/(2m)^2$,

which accounts for the relativistic normalization of the wave functions in field theory. The quantum-mechanical system of two dimensional particles interacting via a δ-type potential develops short-distance divergences and requires a regularization [7,8]. In our case, the cutoff is proportional to m, because at momenta of order m the non-relativistic approximation becomes inadequate.

The binding energy Δm, $\Delta \ll 1$, is determined by the Schrödinger equation for the relative motion wave function:

$$\left(-\frac{1}{m}\,\partial^2 + \Delta m\right)\psi(\mathbf{r}) = \frac{12\lambda}{m^2}\,\psi(0)\delta(\mathbf{r}), \qquad (2)$$

which, after the Fourier transform, gives the consistency condition:

$$1 = \frac{12\lambda}{m^2}\int \frac{d^2p}{(2\pi)^2}\,\frac{1}{p^2/m + \Delta m} = \frac{3\lambda}{\pi m}\ln\frac{\Lambda^2}{\Delta m^2}. \qquad (3)$$

Taking $\Lambda^2 = \kappa m^2$, we get for the binding energy:

$$\Delta = \kappa \exp\left(-\frac{\pi m}{3\lambda}\right). \qquad (4)$$

The constant κ cannot be determined in the approximation used above and requires the inclusion of loop corrections and of the momentum dependence of diagrams b and c. These corrections can be systematically taken into account in the approach based on the Bethe-Salpeter equation, which will be reported elsewhere [9]; here we only quote the result: $\kappa = 4/\sqrt{3}$.

The above discussion tells us that only one bound state of two elementary quanta may exist in the broken phase of the ϕ^4 model in three dimensions. This is in accord with numerical simulations [2,3]. However, a rich spectrum of bound states, with different values of the angular momentum, can be found if we look at the bound states of three or more elementary quanta. Bound states of $n \geq 2$ particles could be studied in principle within the Bethe-Salpeter approach, but even the non-relativistic approximation described above becomes too complicated as the number of particles involved in the bound state increases. Up to our knowledge, the only existing result in the literature is a discussion of the 0^+ bound state of three particles which can be found in [10]. The counterpart of this state is also seen in the numerical simulations [3]. An easier way to understand the qualitative features of the bound states is to study the Ising model in the low-temperature phase far below criticality.

The Ising model at low temperature. To proceed in understanding the structure of the bound state spectrum in the ϕ^4 theory, let us address the same problem in the case of the low temperature phase of the 3d Ising model. The Ising model and the ϕ^4 theory belong to the same universality class. Therefore they should have the same spectrum in the critical limit. Indeed, in [2,3] it was shown in a Monte Carlo study that the two models share the same spectrum of non-perturbative states.

The main advantage of working with the Ising model is that the spectrum can be analyzed in a low temperature expansion of the transfer matrix (see [11,12]). The starting point of this expansion is to ignore the interactions between time-slices. In this approximation, the vectors that correspond to a single configuration on a time-slice become eigenvectors of the transfer matrix. The eigenvalues of the transfer matrix are directly given by the number of frustrated bonds.

In this framework, the bound state of two particles is obtained by flipping two nearby spins. If we flip two spins which are separated by a distance of more than one lattice spacing, the total number of frustrated bonds is exactly twice that of a single particle. On the contrary, if we flip two nearby spins, the number of frustrated bonds is reduced by two. This difference is the binding energy of the bound state. The fact that we are constrained to choose the two spins in nearest neighbor sites is another way to state that the attractive force between the two particles has a very short range. This procedure can be iterated, and one can construct clusters of k nearby flipped spins which have a non-zero binding energy and are related to bound states of higher mass.

It is also possible to select bound states of non-zero angular momentum. These combinations can be constructed by using standard group-theoretical techniques. They are discussed in [13]. Let us only recall here two results which are of interest for the present analysis.

On a (2+1) dimensional lattice the group of rotations and parity reflections is reduced to the dihedral group D^4 which has four one-dimensional and one two-dimensional irreducible representations. The 0^+ state is associated to the trivial one-dimensional irreducible representation. The 2^+ and 2^- states are degenerate and correspond to two other one-dimensional representations. The simplest possible realization of the 2^- state is represented in Fig. 2. At least three flipped spins are needed to create such a state (the 2^+ state could be also realized in a simpler way, but a general theorem forces its mass to be the same of the 2^- one in the continuum limit). Thus we expect that this state should appear as a bound state of at least three elementary quanta. The last one-dimensional irreducible representation corresponds to the 0^- state. The simplest possible representation of this state is reported in Fig. 3 and requires at least four flipped spins. Finally, all the states with odd angular momentum are collected in the two dimensional representation. As in the 2^+,2^- case they are all degenerate in parity.

Let us summarize the pattern of bound states as it emerges from these considerations. With two elementary quanta we may only create a bound state with quantum numbers 0^+. We shall denote it with $0^{+,*}$ to distinguish it from the single particle excitation which has the same quantum numbers. With three quanta we may create a

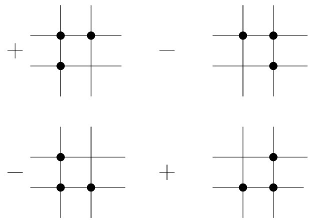

Fig. 2: *Operators for* 2^-.

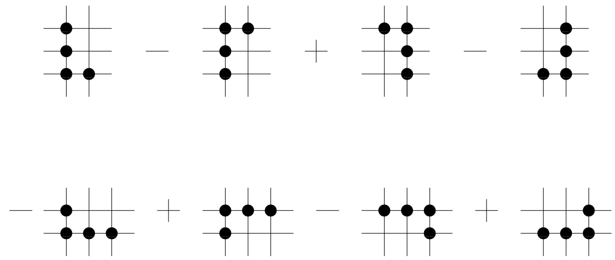

Fig. 3: *Operators for* 0^-.

pair of bound states 2^+ and 2^- and a new 0^+ excitation that we shall call $0^{+,**}$. With four particles we shall have a $2^{\pm,*}$ pair, a $0^{+,***}$ state and a new state with quantum numbers 0^-. With five particles a new pair of states of the type $1^\pm$ appears, and so on.

Let us now make the crucial assumption that the binding energy is always much smaller that the mass of the constituent particles: then the mass of each bound state will be essentially given by the number of particles needed for its formation, minus a small correction given by the binding energy. In this way one obtains a detailed prediction of the qualitative features of the spectrum, based only on the interpretation of the states as bound states and the group-theoretical facts described above.

Numerical simulations [2,3,9,13] show that not only these predictions are fulfilled, but that the same qualitative features of the spectrum survive well beyond the low-temperature regime and into the scaling region. Connected correlators of several composite operators are computed in Monte Carlo simulations and used to extract the spectrum in the various angular momentum channels. The measured masses exactly follow the pattern suggested above. This is a strong indication that the spectrum is indeed made of bound states of the elementary quanta, and that these bound states survive in the continuum limit.

The fact that states with angular momentum 2 are lighter than that with angular momentum 0^- is rather unexpected in standard quantum field theory. However it is a well established feature of the glueball spectrum in (lattice) gauge theories. This is the first hint that the spectrum of bound states of the 3d Ising model has something to do with the glueballs of gauge theory. The reason for this is obviously the duality between spin model and gauge model, that we will now discuss.

Duality. Duality is usually expressed as an exact equality between partition functions in infinite volume, hence in principle it does not automatically implies that the two theories must have the same spectrum. However it can be shown that duality holds not only in the thermodynamic limit *but also for finite lattices*. This correspondence is not trivial and requires a careful analysis of the boundary conditions of the two models [9]. Since the approach to the thermodynamic limit of the finite volume partition function is driven by the full spectrum of excited states of the theory, the finite volume duality implies that the spectra of the two models must coincide. In particular the bound state of quantum numbers J^P of the Ising spin model coincides (hence has exactly the same mass) with the J^P glueball of the gauge Ising model. This identification has two interesting consequences. The first one is that the Bethe-Salpeter approach to the calculation of bound states in ϕ^4 theory, described above, becomes an analytical tool to evaluate the masses of the first states of the glueball spectrum of the gauge Ising model. In principle (apart from technical difficulties) this could be extended to the whole glueball spectrum, and represents a powerful alternative to the Isgur-Paton model, which in the case of the Ising gauge model gives rather poor result [13]. A second, more important consequence of this identification is that it gives a possible explanation for a peculiar degeneracy observed in the Monte Carlo estimates of the glueball masses in the 3d gauge Ising model [13] for which no alternative explanation exists. This intriguing feature of the spectrum can be immediately appreciated by looking at Tab. 1, (data taken from [13,14]). In Tab. 1 the asterisks denote the radial excitations, thus 0^+ is the lowest state in the family with quantum numbers 0^+, $0^{+,*}$ the next one and $0^{+,**}$ the third one. 0^+ is related by duality to the single-particle state of the 3d Ising model, $0^{+,*}$ to the first bound state and so on. The degeneracy involves the pairs $(0^{+,**}, 2^\pm)$, $(0^-, 2^{\pm,*})$, $(0^{-,*}, 1^\pm)$ (the last one is only roughly established, it holds within the errors). Let us stress that this degeneracy has no obvious physical reason. The only one which we would expect on physical grounds is the one between J^+ and J^- states (for $J \neq 0$) (see [13] for a discussion of this point) which is indeed present and has been already taken into account in Tab. 1. Moreover it *is not* explained by the Isgur-Paton model (last column of Tab. 1). This degeneracy seems to be a rather deep phenomenon since it is also present

in the glueball spectrum of the SU(2) model in (2+1) dimensions. In the third column of Tab. 1 we report the data on SU(2) obtained by Teper [14] (the underlined values are our extrapolations of the finite-β values reported in [14]). One can easily see that the same pattern of degeneracy is present both in the SU(2) and in the Ising gauge spectra. On the contrary, all these degeneracies seem to be lost in the $SU(N)$, $(N > 2)$ case (see the data reported in [14]).

This degeneracy is well explained by the interpretation of the glueballs as bound states of the dual spin model: the degenerate glueball states are simply bound states of the *same number* n_c of constituent particles, namely $n_c = 3, 4, 5$ respectively for the $(0^{+,**}, 2^{\pm})$, $(0^{-}, 2^{\pm,*})$ and $(0^{-,*}, 1^{\pm})$ degeneracies. In fact, according to the assumption stated above, the major contribution to the mass of the bound state is given by the number of elementary quanta involved. The dependence on the various quantum numbers is encoded in the binding energies Δ which however give only a small correction to the mass. This results in the approximate degeneracies observed in the simulations. Notice that we do not expect to have *exact* degeneracies, since there is no reason to expect the binding energy to be exactly the same for different bound states.

TABLE I. *Comparison between the Ising, SU(2) and Isgur-Paton spectra. The masses are measured in units of the string tension.*

J^P	Ising	SU(2)	IP
0^{+}	3.08(3)	4.718(43)	2.00
$0^{+,*}$	5.80(4)	6.83(10)	5.94
$0^{+,**}$	7.97(11)	8.15(15)	8.35
$2^{\pm}$	7.98(8)	7.84(14)	6.36
$2^{\pm,*}$	9.95(20)	$\underline{9.30(50)}$	8.76
0^{-}	10.0(5)	$\underline{9.95(32)}$	13.82
$0^{-,*}$	13.8(6)	$\underline{11.30(80)}$	15.05
$(1)^{\pm}$	12.7(5)	$\underline{10.75(50)}$	8.04

Conclusions. Our analysis shows that bound states are very likely to exist in the broken-symmetry phase of 3d ϕ^4 and Ising models. Their existence can be inferred both from the Bethe-Salpeter equation of the field theory and the strong-coupling analysis of the spin model, and is strongly confirmed by numerical simulations.

Duality allows one to apply the same analysis to the glueball spectrum of the 3d Ising gauge model, which exactly coincides with the one of the spin model. The interpretation of the latter as a spectrum of bound states provides a natural explanation for several features of the glueball spectrum, such as its peculiar dependence on the angular momentum and its characteristic degeneracies.

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